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Limiting subgradients of minimal time functions in Banach spaces

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Abstract The paper mostly concerns the study of generalized differential properties of the so-called *minimal time functions* associated, in particular, with constant dynamics and arbitrary closed target sets in control theory. Functions of this type play a significant role in many aspects of optimization, control theory, and Hamilton–Jacobi partial differential equations. We pay the main attention to computing and estimating limiting subgradients of the minimal value functions and to deriving the corresponding relations for Fréchet type ε-subgradients in arbitrary Banach spaces.

Keywords Variational analysis · Optimization and optimal control · Hamilton–Jacobi equations · Minimal time functions · Minkowski functions · Generalized differentiation · Banach spaces

Mathematics Subject Classification (2000) 49J52 · 49J53 · 90C31

1 Introduction

This paper is devoted to the study of subdifferential properties of a broad class of the so-called *minimal time functions*, which play a highly important role in many aspects of variational analysis, optimization, control theory, Hamilton–Jacobi partial differential equations, approximation theory, etc.; the reader can find more information and discussions in [4,6–10,19,20] and the references therein. To the best of our knowledge, a systematic study of this class

Dedicated to Franco Giannessi in honor of his 75th birthday.

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of functions was started by Bardi [1] who characterized a minimal time function in control theory as a viscosity solution to a Hamilton–Jacobi equation.

The main attention of this paper is paid to the minimal time functions defined by

$$\tau_F(x;\Omega) := \inf_{w\in\Omega} p_F(w-x), \quad x \in X, \tag{1.1}$$

where (in our *standing assumptions*) $F \subset X$ is a closed, convex, and bounded subset of a Banach space X with $0 \in \operatorname{int} F$, where the set $\Omega \subset X$ is a closed (while nonconvex in general) subset of X, and where

$$p_F(u) := \inf\{t > 0 \mid t^{-1}u \in F\}, \quad u \in X, \tag{1.2}$$

is the classical *Minkowski function* (gauge) of F; see e.g., [16]. The minimal function (1.1) can be associated with a *control system* involving the *constant dynamics* $\dot{x}(t) \in F$, where the *velocity set* F is independent of (x, t), and where the *target set* Ω is arbitrary closed. On the other hand, the minimal time function $\tau_F(\cdot;\Omega)$ can be viewed as the *marginal/value function* in the corresponding *parametric constrained optimization* problem which objective is described by the Minkowski function (1.2) also generated by the infimum operation. In particular, if $F = I\!B$ is the closed unit ball $I\!B \subset X$, then we obviously have $p_F(u) = \|u\|$ while (1.1) reduces to the *distance function* of the set Ω given by

$$d(x;\Omega) := \inf_{w\in\Omega} \|x-w\|, \quad x \in X. \tag{1.3}$$

There are significant differences between the distance function (1.3) and the minimal time function (1.1) studied in this paper. In particular, the closed and convex set $F \subset X$ generating (1.1) and (1.2) may be *asymmetric*, in contrast to the ball $I\!B$ generating (1.3).

Observe that all the three functions (1.1)–(1.3) are *nonsmooth*, and hence require tools of *generalized differentiation* for their study. Furthermore, while the Minkowski function (1.2) is convex under the assumptions made, the minimal time function (1.1) and its distance specification (1.3) are generally *nonconvex* unless the target set Ω is assumed to be convex, which is not the case in this paper.

Subdifferential properties of the distance function (1.3) have been well investigated and applied in many publications; see e.g., [2,3,5,11–14,17] and the references therein. Much less has been done for the minimal time function (1.1). We mention the papers [6,7,19,20] that contain estimating and computing proximal subgradients of (1.1) in finite-dimensional and Hilbert spaces with some applications to control theory while [7] establishes certain results of this type for Fréchet subgradients in Hilbert spaces. Directional derivative properties of (1.1) with applications to well-posedness and approximation problems are given [8,10]. Finally, the more recent study [9] is devoted to deriving formulas for the evaluation of the proximal, Fréchet, and Clarke subdifferentials of the minimal time function (1.1) in the arbitrary Banach space setting.

In this paper we mainly focus on evaluating another subdifferential of (1.1), which is the *smallest robust* subdifferential satisfying certain mandatory requirements for the general class of extended-real-valued functions and is widely spread in variational analysis and its applications under the names of *basic/limiting/Mordukhovich subdifferential*; see the books [11,12] for a systematic study and applications of this subdifferential and the normal cone/coderivative constructions associated with it and also the books [2,5,17,18] for related and additional material. Among the major advantages of the latter subdifferential are extended *calculus rules* partly developed in general Banach spaces (see e.g., [11, Chapt. 1]), which are *comprehensive* [11, Chapt. 3] when the space in question is Asplund, i.e., each of its separable subspace has a separable dual; in particular, for reflexive spaces.

To best of our knowledge, there are no results available for limiting subgradients of the minimal time function even in finite-dimensional spaces. Our setting in this paper is *arbitrary Banach*. To proceed with calculating and estimating the limiting subdifferential, we establish first the corresponding results for ε-*subgradients* of the Fréchet type for the minimal time function in general Banach spaces; some of the latter results are fully new while the others are extensions and clarifications of those obtained in [9] for the case of Fréchet subgradients ($\varepsilon = 0$). In particular, we are able to fill the gap in the proof of [9, Theorem 4.2] for Fréchet subgradients in Banach spaces; see Sect. 4.

The results derived in this paper can be viewed as extensions of our previous developments [13,14] for the distance function (1.3); see also [11, Sect. 1.3.3]. Similarly to the distance function, we pay the main attention here to evaluating subgradients of (1.1) at *out-of set* points $x \notin \Omega$, which is essentially more involved in comparison with the *in-set* case $x \in \Omega$. It is worth mentioning that, although the minimal time function under consideration belongs to the broad infimum-generated class of *marginal functions*

$$\mu(x) := \inf_{w \in \Omega(x)} \varphi(x, w), \quad x \in X, \tag{1.4}$$

particularly studied in the recent publications [11,15], *none* of the results obtained in this paper can be derived from those available for general marginal functions of type (1.4). Indeed, the "upper subdifferential" results for evaluating Fréchet-type subgradients of (1.4) given in [15] are simply not applied to (1.1) due to nonsmoothness and convexity of the Minkowski function (1.2). The corresponding results of [11,15] on evaluating limiting subgradients of the marginal function (1.4) are applied only to the "in-set" case of (1.1) in the Asplund space setting providing generally rougher upper estimates in comparison with the results of this paper that also contain precise/equality formulas. In short, *all* the results established below are due to the specific Minkowski form of the cost function in (1.4), which is a significant while reasonable extension of the norm function used in (1.3).

The rest of the paper is *organized as follows*. In Sect. 2, we present basic definitions and preliminaries broadly used in formulations and proofs of the main results of the paper. Section 3 is devoted to evaluating ε-subgradients and limiting subgradients of the minimal time function (1.1) at in-set and out-of-set point of the target set Ω in (1.1) via the corresponding normals to Ω. In Sect. 4, we establish further relations between the afore-mentioned subgradients of (1.1) at out-of-set points and normals to appropriate enlargements of Ω. The major result of this section employs the recent construction of the so-called right-sided/outer limiting subdifferential introduced in [13]. The final Sect. 5 presents new relations between the ε-subdifferential and the limiting subdifferential of the (nonconvex) minimal time function under consideration and the convex subdifferential and its ε-enlargement for the generated Minkowski function (1.2).

Throughout the paper, we use standard *notation* of variational analysis and generalized differentiation; see e.g., [11]. Unless otherwise stated, the space X in question is *Banach* with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between X and its topological dual x^*. As usual, the symbol $x_k \to \bar{x}$ stands for the norm convergence in X while $x_k^* \xrightarrow{w^*} x^*$, $k \in I\!N := \{1, 2, \ldots\}$, signifies the sequential weak* convergence in the dual space X^*. Given a set-valued mapping $G \colon X \rightrightarrows X^*$, we denote

$$\mathop{\text{Lim sup}}_{x \to \bar{x}} G(x) : = \Big\{ x^* \in X^* \,|\, \exists \text{ sequences } x_k \to \bar{x}, \;\; x_k^* \xrightarrow{w^*} x^* \text{ as } k \to \infty \text{ with } x_k^* \in G(x_k) \quad \text{for all} \quad k \in I\!N \Big\} \tag{1.5}$$

the *sequential* Painlevé-Kuratowski upper/outer limit of G as $x \to \bar{x}$. If no confusion arises, the symbol $x \xrightarrow{\Omega} x$ means that $x \to \bar{x}$ with $x \in \Omega$ for a set Ω, while $x \xrightarrow{\varphi} \bar{x}$ indicates that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$ for an extended-real-valued function $\varphi\colon X \to \overline{I\!R} := (-\infty, \infty]$.

2 Basic definitions and preliminaries

In this section we first present, mainly following [11], basic constructions and properties from variational analysis and generalized differentiation broadly used in the paper. Then we formulate some preliminary properties of the Minkowski and minimal time functions, which are needed in what follows and can be found in [9,16].

Let Ω be a nonempty subset in a Banach space X. Given any $\varepsilon \geq 0$, the (convex) set of ε-*normals* to Ω at $\bar{x} \in \Omega$ is defined by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \,\middle|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\} \tag{2.1}$$

and by $\widehat{N}_\varepsilon(\bar{x}; \Omega) = \emptyset$ if $\bar{x} \notin \Omega$. When $\varepsilon = 0$ in (2.1), the set $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$ is a convex cone called the *Fréchet normal cone* to Ω at $\bar{x}$. The sequential outer limit

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x; \Omega) \tag{2.2}$$

of (2.1) is known as the *basic/limiting/Mordukhovich normal cone* to Ω at $\bar{x} \in \Omega$. By (1.5), limiting normals from (2.2) can be described as follows: $x^* \in N(\bar{x}; \Omega)$ *if and only if* there are sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ for all $k \in I\!N$. By [11, Theorem 2.35], we can equivalently let $\varepsilon = 0$ in (2.2) if the set Ω is locally closed around $\bar{x}$ and if the space X is Asplund. Note that, in contrast to (2.1) and the vast majority of other known normal cone constructions in nonsmooth analysis (including, in particular, the proximal and Clarke normal cones), our basic normal cone (2.2) is often *nonconvex* (even for simple sets in $I\!R^2$), while it and related subdifferential and coderivative constructions for functions and mappings admit well-developed pointwise calculus rules that are essentially better than for their convex-valued counterparts; see [11,12] and also [2,17,18] with the references and commentaries therein for more details and discussions. Let us emphasize that the afore-mentioned calculus is largely based on *variational/extremal principles* of variational analysis.

Given and extended-real-valued function $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$, define the ε-*subdifferential* of φ at this point by

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) := \left\{ x^* \in X^* \,\middle|\, \liminf_{x \xrightarrow{\varphi} \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}, \tag{2.3}$$

which reduces to the *Fréchet subdifferential* $\widehat{\partial}\varphi(\bar{x}) := \widehat{\partial}_0\varphi(\bar{x})$ of φ at $\bar{x}$ for $\varepsilon = 0$. If φ is *convex*, the ε-subdifferential (2.3) reduces to

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) = \left\{ x^* \in X^* \,|\, \langle x^*, x - \bar{x}\rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon\|x - \bar{x}\| \quad \text{for all} \quad x \in X \right\}, \tag{2.4}$$

which goes back to the classical subdifferential of convex analysis as $\varepsilon = 0$. Similarly to (2.2), the *basic/limiting/Mordukhovich subdifferential* of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \limsup_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{\partial}_\varepsilon \varphi(x), \tag{2.5}$$

where we can equivalently put $\varepsilon = 0$ if φ is lower semicontinuous (l.s.c.) around $\bar{x}$ and if the space X is Asplund. It is worth mentioning the equivalent *geometric description* of the basic subdifferential (2.5) via the normal cone (2.2) valid in arbitrary Banach spaces:

$$\partial\varphi(\bar{x}) = \left\{x^* \in X^* \mid (x^*, -1) \in N\left((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi\right)\right\},$$

where $\operatorname{epi}\varphi := \{(x, \mu) \in X \times I\!R \mid \mu \geq \varphi(x)\}$ is the *epigraph* of the function φ.

Considering further a nonempty set $F \subset X$, recall that

$$F^\circ := \left\{x^* \in X^* \mid \langle x^*, v\rangle \leq 1 \quad \text{for all} \quad v \in F\right\}$$

is the *polar* to F, which is always convex (even when F is nonconvex) and weak* closed subset of the dual space X^* with $0 \in F^\circ$. Denote

$$\|F\| := \sup\{\|v\| \mid v \in F\} \quad \text{and thus} \quad \|F^\circ\| := \sup\left\{\|x^*\| \mid x^* \in F^\circ\right\}.$$

The following proposition summarizes some well-known properties of the Minkowski function (1.2), under the standing assumptions formulated in Sect. 1, which are crucial for the main results obtained in this paper; see e.g., [16, Sect. 1].

Proposition 2.1 (properties of the Minkowski function). *Let F be a bounded, closed, and convex set with* int $F \neq \emptyset$. *Then the following hold for the Minkowski function* (1.2)*:*

(i) *p_F is finite, positively homogeneous, and subadditive on X.*
(ii) *We have the representations*

$$p_{F^\circ}(x^*) = \sup_{v \in F}\langle x^*, v\rangle \quad \text{and} \quad p_F(v) = \sup_{x^* \in F^\circ}\langle x^*, v\rangle.$$

(iii) *p_F is Lipschitz continuous on X with Lipschitz constant $\|F^\circ\|$.*

The next proposition presents two properties of the minimal time function (1.1) used in what follows. They are implied by properties (i) and (ii) from Proposition 2.1; see [9, Lemmas 3.1, 3.2] for more details.

Proposition 2.2 (properties of the minimal time function). *Let F satisfy the assumptions of Proposition 2.1, and let Ω be a closed subset of X. Then we have:*

(i) *$\tau_F(x; \Omega) - \tau_F(y; \Omega) \leq p_F(y - x)$ for all $x, y \in X$.*
(ii) *$\tau_F(\cdot; \Omega)$ is Lipschitz continuous on X with Lipschitz constant $\|F^\circ\|$.*

3 Subgradients of minimal time functions via normals to target sets

In this section, we obtain various representations of ε-subgradients and limiting subgradients of the minimal time function $\tau_F(\cdot; \Omega)$ at in-set points $\bar{x} \in \Omega$ and at out-of-set ones $\bar{x} \notin \Omega$ via the corresponding normals to the target set Ω and the polar to the Minkowski function of the velocity set F. We start with the following preliminary result used in the sequel.

Lemma 3.1 (minimal time functions at intermediate points). *Define the* MINIMUM SET (*or "generalized projection"*) *for the minimal time function* (1.1) *by*

$$M_F(x;\Omega) := \{w \in \Omega|\ \tau_F(x;\Omega) = p_F(w-x)\}\,. \tag{3.1}$$

Then for every $\bar{x} \notin \Omega$, for every $\bar{w} \in M(\bar{x})$, and for every $t \in (0,1]$ we have

$$\tau_F\,(t\bar{w} + (1-t)\bar{x};\Omega) = (1-t)\tau_F(\bar{x};\Omega), \tag{3.2}$$

which implies, in particular, that

$$\bar{w} \in M_F\,(t\bar{w} + (1-t)\bar{x};\Omega)\,,\quad 0 < t \le 1. \tag{3.3}$$

Proof It is easy to observe from the definitions and the choice of $\bar{w} \in M(\bar{x})$ that

$$\begin{aligned}\tau_F\,(t\bar{w} + (1-t)\bar{x};\Omega) &\le p_F\,(\bar{w} - t\bar{w} - (1-t)\bar{x})\\ &= p_F\,((1-t)(\bar{w}-\bar{x}))\\ &= (1-t)p_F(\bar{w}-\bar{x}) = (1-t)\tau_F(\bar{x};\Omega).\end{aligned}$$

Denoting $x_t := t\bar{w} + (1-t)\bar{x}$ as $t \in (0,1]$ and selecting $w_k \in \Omega$ for each $k \in \mathbb{N}$, such that

$$p_F(w_k - x_t) \to \tau_F(x_t;\Omega) \quad \text{as} \quad k \to \infty,$$

we get by the subadditivity property of p_F from Proposition 2.1 (i) that

$$\begin{aligned}p_F(w_k - x_t) &= p_F\,(w_k - \bar{x} - t(\bar{w}-\bar{x})) \ge p_F(w_k - \bar{x}) - p_F\,(t(\bar{w}-\bar{x}))\\ &\ge \tau_F(\bar{x};\Omega) - t\tau_F(\bar{x};\Omega) = (1-t)\tau_F(\bar{x};\Omega),\quad k \in \mathbb{N},\quad 0 < t \le 1.\end{aligned}$$

This justifies equality (3.2) by letting $k \to \infty$, which easily implies inclusion (3.3) and thus completes the proof of the lemma. □

The next result establishes *two-sided* estimates for the Minkowski function (1.2) of the polar F° to the velocity set at ε-subgradients of the minimal time function (1.1) calculated at *out-of-set* points. It is certainly of independent interest while providing useful information for subsequent subgradient evaluations of the minimal time function.

Proposition 3.2 (relations between ε-subgradients of minimal time functions at out-of-set points and behavior of the corresponding polar Minkowski function). *For every $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x};\Omega)$ with $\bar{x} \notin \Omega$ and $\varepsilon \ge 0$ we have the lower and upper estimates*

$$1 - \varepsilon\|F\| \le p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|. \tag{3.4}$$

Proof Fix any $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x};\Omega)$ justify first the *upper estimate* in (3.4). Given any $\eta > 0$, find $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \tau_F(x;\Omega) - \tau_F(\bar{x};\Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{as} \quad x \in \bar{x} + \delta\mathbb{B}. \tag{3.5}$$

For every $v \in X$ choose $t > 0$ so small that $\bar{x} - tv \in \bar{x} + \delta\mathbb{B}$. Then

$$\begin{aligned}t\langle x^*, -v\rangle &\le \tau_F(\bar{x} - tv;\Omega) - \tau_F(\bar{x};\Omega) + (\varepsilon+\eta)t\|v\|\\ &\le p_F\,(\bar{x} - (\bar{x} - tv)) + (\varepsilon+\eta)t\|v\|\\ &\le tp_F(v) + (\varepsilon+\eta)t\|v\|.\end{aligned}$$

The latter implies the estimate

$$\langle x^*, -v\rangle \le 1 + (\varepsilon+\eta)\|v\| \quad \text{for all} \quad v \in F$$

by the well-known representation

$$F = \{x \in X |\ p_F(x) \le 1\}.$$

Thus $\sup_{v \in F}\langle x^*, -v\rangle \le 1 + \varepsilon\|F\|$, and we get $p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|$ by Proposition 2.1 (ii), which justifies the upper estimate in (3.4).

To prove the *lower estimate* in (3.4), for any $t > 0$ choose $w_t \in \Omega$ such that

$$p_F(w_t - \bar{x}) < \tau_F(\bar{x}) + t^2.$$

Noticing that $(w_t - \bar{x})/p_F(w_t - \bar{x}) \in F$ and denoting $x_t := \bar{x} + t(w_t - \bar{x})$, we have

$$\|x_t - \bar{x}\| = t\|w_t - \bar{x}\| \le t p_F(w_t - \bar{x})||F|| \le \delta$$

whenever t is sufficiently small. The latter implies the relations

$$\begin{aligned} t\langle x^*, w_t - \bar{x}\rangle &\le \tau_F(x_t; \Omega) - \tau_F(\bar{x}; F) + (\varepsilon + \eta)t\|w_t - \bar{x}\| \\ &\le \tau_F(x_t; \Omega) - p_F(w_t - \bar{x}) + t^2 + (\varepsilon + \eta)t\|w_t - \bar{x}\| \\ &\le p_F(w_t - x_t) - p_F(w_t - \bar{x}) + t^2 + (\varepsilon + \eta)t\|w_t - \bar{x}\| \\ &= p_F\left((1 - t)(w_t - \bar{x})\right) - p_F(w_t - \bar{x}) + t^2 + (\varepsilon + \eta)t\|w_t - \bar{x}\| \\ &= -t p_F(w_t - \bar{x}) + t^2 + (\varepsilon + \eta)t p_F(w_t - \bar{x})\|F\|. \end{aligned}$$

Hence we have the estimate

$$\left\langle x^*, \frac{w_t - \bar{x}}{p_F(w_t - \bar{x})}\right\rangle \le -1 + \frac{t}{p_F(w_t - \bar{x})} + (\varepsilon + \eta)\|F\|.$$

Define further $v := (w_t - \bar{x})/p_F(w_t - \bar{x}) \in F$ and get

$$1 - \frac{t}{p_F(w_t - \bar{x})} - (\varepsilon + \eta)\|F\| \le \langle -x^*, v\rangle \le \sup_{v \in F}\langle -x^*, v\rangle = p_{F^\circ}(-x^*). \tag{3.6}$$

Letting $t \to 0$ and then $\eta \to 0$ in (3.6), we arrive at $1 - \varepsilon\|F\| \le p_{F^\circ}(-x^*)$, which gives the lower estimate in (3.4) and complete the proof of the theorem. □

The next result establishes a relation between the ε-subdifferential of the minimal time function at *out-of-set* and *intermediate* points. Denote by

$$S^*_\varepsilon := \left\{x^* \in X^* |\ 1 - \varepsilon\|F\| \le p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|\right\}, \quad \varepsilon \ge 0, \tag{3.7}$$

the collection of dual vectors satisfying the estimates in (3.4).

Proposition 3.3 (relation between ε-subgradients of minimal time functions at out-of-set and intermediate points). *Let $\varepsilon \ge 0$, $\bar{x} \notin \Omega$, and $\bar{w} \in M(\bar{x})$ for the minimum set defined in* (3.1). *Then we have the inclusion*

$$\widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega) \subset \widehat{\partial}_\varepsilon \tau_F\left(t\bar{w} + (1 - t)\bar{x}; \Omega\right) \cap S^*_\varepsilon \quad \text{for all} \quad t \in (0, 1], \tag{3.8}$$

where the set S^*_ε is defined in (3.7).

Proof Picking any $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega)$, we get from Proposition 3.2 that $x^* \in S^*_\varepsilon$. To justify (3.8), it remains to show that

$$x^* \in \widehat{\partial}_\varepsilon \tau_F\left(t\bar{w} + (1 - t)\bar{x}; \Omega\right) \quad \text{for all} \quad t \in (0, 1]. \tag{3.9}$$

It follows from Definition (2.3) of ε-subgradients that for any $\eta > 0$ there is $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{whenever} \quad \|x - \bar{x}\| \le \delta.$$

Considering again $x_t = t\bar{w} + (1-t)\bar{x}$ as $0 < t \le 1$, we have that

$$\|x - t(\bar{w} - \bar{x}) - \bar{x}\| \le \delta \quad \text{for all} \quad x \in X \;\; \text{with} \;\; \|x - x_t\| \le \delta.$$

Then the results from Proposition 2.2 (i) and Lemma 3.1 allow us to conclude that

$$\begin{aligned}\langle x^*, x - x_t\rangle &\le \tau_F\,(x - t(\bar{w} - \bar{x}); \Omega) - \tau_F(\bar{x}; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \\ &\le \tau_F(x; \Omega) + t p_F(\bar{w} - \bar{x}) - \tau_F(\bar{x}; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \\ &= \tau_F(x; \Omega) - (1-t)\tau_F(\bar{x}; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \\ &= \tau_F(x; \Omega) - \tau_F(x_t; \Omega) + (\varepsilon + \eta)\|x - x_t\|,\end{aligned}$$

which implies (3.9) by (2.3) and thus completes the proof of the proposition. □

Now we consider *two-sided* estimates for the ε-subdifferential of the minimal function via ε-normals to the target set at *in-set* points. The next theorem extends the result of [9, Theorem 4.1] obtained for $\varepsilon = 0$.

Theorem 3.4 (relations between ε-subgradients of minimal time functions and ε-normals to target sets at in-set points). *Let $\bar{x} \in \Omega$ for the minimal time function* (1.1). *Then for any $\varepsilon \ge 0$ we have the relations*

$$\widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega) \subset \widehat{N}_\varepsilon(\bar{x}; \Omega) \cap \left\{x^* \in X^* \mid p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|\right\} \subset \widehat{\partial}_{\alpha\varepsilon}\tau_F(\bar{x}; \Omega) \tag{3.10}$$

with the perturbation parameter $\alpha > 0$ in (3.10) *defined by $\alpha := 2\|F\| \cdot \|F^\circ\| + 1$.*

Proof Fix $\varepsilon \ge 0$ and pick any $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega)$. By the subdifferential Definition (2.3) for the minimal time function (1.1) and its description in (3.5) we immediately have that for $\eta > 0$ there is $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le (\eta + \varepsilon)\|x - \bar{x}\| \quad \text{whenever} \quad x \in \Omega \cap (\bar{x} + \delta I\!B),$$

since $\tau_F(x) = \tau_F(\bar{x}) = 0$ when $x \in \Omega$. Furthermore, it follows from the proof of the first part of Proposition 3.2 (which works for both case of $\bar{x} \in \Omega$ and $\bar{x} \notin \Omega$) that

$$p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|. \tag{3.11}$$

Thus we get the first inclusion in (3.10) for any $\varepsilon \ge 0$.

To justify the second inclusion in (3.10) with each fixed $\varepsilon \ge 0$, take any $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega)$ satisfying (3.11). By definition (2.1) of ε-normals, for a given $\eta > 0$ we find $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{whenever} \quad \|x - \bar{x}\| \le \delta \;\; \text{and} \;\; x \in \Omega.$$

As mentioned above, this can be equivalently written in the form of

$$\langle x^*, x - \bar{x}\rangle \le \tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for all} \quad x \in \Omega \cap (\bar{x} + \delta I\!B). \tag{3.12}$$

To get further a counterpart of (3.12) at out-of-set points, define

$$\widetilde{\delta} := \frac{\delta}{2 + 2\|F^\circ\| \cdot \|F\|}$$

and fix $x \notin \Omega$ with $\|x - \bar{x}\| \le \widetilde{\delta}$. Taking into account the Lipschitz property of $p_F(\cdot)$ from Proposition 2.1 (iii), we get the estimates

$$\tau_F(x; \Omega) \le p_F(\bar{x} - x) \le \|F^\circ\| \cdot \|\bar{x} - x\| < \|F^\circ\|\widetilde{\delta}.$$

Choose now $w_k \in \Omega$ such that $p_F(w_k - x) < \|F^\circ\|\widetilde{\delta}$ for all $k \in \mathbb{N}$ and $p_F(w_k - x) \to \tau_F(x; \Omega)$ as $k \to \infty$. Then

$$\|w_k - x\| \le \|F\| p_F(w_k - x) \le \|F\| \cdot \|F^\circ\|\widetilde{\delta} < \delta/2, \quad k \in \mathbb{N}, \tag{3.13}$$

which implies that $\|w_k - \bar{x}\| \le \|w_k - x\| + \|x - \bar{x}\| < \delta$ for all $k \in \mathbb{N}$. Denoting

$$v_k := \frac{w_k - x}{p(w_k - x)} \in F$$

and taking into account the choice of x^*, x, and w_k, we have the following estimates:

$$\begin{aligned}
\langle x^*, x - \bar{x}\rangle &\le \langle x^*, x - w_k\rangle + \langle x^*, w_k - \bar{x}\rangle \\
&\le p(w_k - x)\left\langle -x^*, \frac{w_k - x}{p(w_k - x)}\right\rangle + (\varepsilon + \eta)\|w_k - \bar{x}\| \\
&\le p(w_k - x)\langle -x^*, v_k\rangle + (\varepsilon + \eta)\|w_k - x\| + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le p(w_k - x)\langle -x^*, v_k\rangle + (\varepsilon + \eta)\|F\| p(w_k - x) + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le p(w_k - x)\left(1 + (2\varepsilon + \eta)\|F\|\right) + (\varepsilon + \eta)\|x - \bar{x}\|.
\end{aligned}$$

Passing to the limit as $k \to \infty$ in the latter estimate and using (3.13) as well as the convergence $p_F(w_k - x) \to \tau_F(x; \Omega)$, we arrive at

$$\begin{aligned}
\langle x^*, x - \bar{x}\rangle &\le \tau_F(x; \Omega)\left(1 + (2\varepsilon + \eta)\|F\|\right) + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le \tau_F(x; \Omega) + (2\varepsilon + \eta)\|F\|\tau_F(x; \Omega) + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le \tau_F(x; \Omega) + (2\varepsilon + \eta)\|F\| p_F(\bar{x} - x) + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le \tau_F(x; \Omega) + (2\varepsilon + \eta)\|F\| \cdot \|F^\circ\| \cdot \|x - \bar{x}\| + (\varepsilon + \eta)\|x - \bar{x}\| \\
&\le \tau_F(x; \Omega) + \left[\varepsilon\left(2\|F\| \cdot \|F^\circ\| + 1\right) + \eta\left(\|F\| \cdot \|F^\circ\| + 1\right)\right]\|x - \bar{x}\|
\end{aligned}$$

whenever $\|x - \bar{x}\| < \widetilde{\delta}$. Since $\eta > 0$ was chosen arbitrarily and (3.12) was justified, this gives the subgradient inclusion

$$x^* \in \widehat{\partial}_{\alpha\varepsilon}\tau_F(\bar{x}; \Omega) \quad \text{with} \quad \alpha = 2\|F\| \cdot \|F^\circ\| + 1$$

and thus completes the proof of the theorem. □

Let us continue with a result establishing a certain relation between ε-subgradients of minimal time functions at *out-of-set* points of target sets and *perturbed* ε-normals at some *perturbed generalized projections* on the sets in question. The proof is strongly based on *variational/perturbation techniques* of variational analysis.

Theorem 3.5 (ε-subgradients of minimal time functions at out-of-set points via extended normals to perturbed generalized projections on target sets). *Let $\bar{x} \notin \Omega$ for the target set Ω in the minimal time function* (1.1). *Then for every $\varepsilon \ge 0$, every $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega)$, and every $\eta > 0$ there is $\bar{w} \in \Omega$ such that*

$$x^* \in \widehat{N}_{\varepsilon+\eta}(\bar{w}; \Omega) \quad \textit{and} \quad \|\bar{x} - \bar{w}\| \le \|F\|\tau_F(\bar{x}; \Omega) + \eta. \tag{3.14}$$

Proof Fix $\varepsilon \ge 0$, $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega)$, and $\eta > 0$. By the ε-subdifferential definition (2.3) there is $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega) + \left(\varepsilon + \frac{\eta}{2}\right)\|x - \bar{x}\| \quad \text{for all} \quad x \in \bar{x} + \delta\mathbb{B}. \tag{3.15}$$

It follows directly from construction (1.1) of the minimal time function that there exists $\widetilde{w} \in \Omega$ satisfying the inequality

$$p_F(\widetilde{w} - \bar{x}) < \tau_F(\bar{x}; \Omega) + \widetilde{\eta}^2 \quad \text{with} \quad \widetilde{\eta} := \min\left\{\frac{\delta}{2}, \frac{\eta}{2 + \|F\|}, 1\right\}. \tag{3.16}$$

Combining (3.15) and (3.16) allows us to conclude that for any $w \in \Omega \cap (\widetilde{w} + \delta I\!B)$ we have

$$\begin{aligned}\langle x^*, w - \widetilde{w}\rangle &\leq \tau_F(w - \widetilde{w} + \bar{x}; \Omega) - \tau_F(\bar{x}; \Omega) + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| \\ &\leq \tau_F(w - \widetilde{w} + \bar{x}; \Omega) - p_F(\widetilde{w} - \bar{x}) + \widetilde{\eta}^2 + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| \\ &\leq \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| + \widetilde{\eta}^2.\end{aligned}$$

Consider now the *complete metric space* $E := \Omega \cap (\widetilde{w} + \delta I\!B)$ and a continuous function $\varphi\colon E \to I\!R$ defined by

$$\varphi(w) := -\langle x^*, w - \widetilde{w}\rangle + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| + \widetilde{\eta}^2, \quad w \in E.$$

It is easy to observe from the constructions of $\widetilde{w}$ and φ that

$$\varphi(\widetilde{w}) \leq \inf_{w \in E} \varphi(w) + \widetilde{\eta}^2.$$

Applying then the *Ekeland variational principle*, we find $\bar{w} \in E$ such that $\|\widetilde{w} - \bar{w}\| < \widetilde{\eta}$ and

$$\varphi(\bar{w}) \leq \varphi(w) + \widetilde{\eta}\|w - \bar{w}\| \quad \text{for all} \quad w \in E,$$

which readily gives the estimate

$$-\langle x^*, \bar{w} - \widetilde{w}\rangle + \left(\varepsilon + \frac{\eta}{2}\right)\|\bar{w} - \widetilde{w}\| + \widetilde{\eta}^2 \leq -\langle x^*, w - \widetilde{w}\rangle + \left(\varepsilon + \frac{\eta}{2}\right)\|w - \widetilde{w}\| + \widetilde{\eta}^2 + \widetilde{\eta}\|w - \bar{w}\|$$

for all $w \in E$. The latter yields that

$$\langle x^*, w - \bar{w}\rangle \leq \left(\varepsilon + \frac{\eta}{2} + \widetilde{\eta}\right)\|w - \bar{w}\| \leq (\varepsilon + \eta)\|w - \bar{w}\|.$$

It is easily implied by $\|w - \bar{w}\| < \widetilde{\eta}$ that

$$\|w - \widetilde{w}\| \leq \|w - \bar{w}\| + \|\bar{w} - \widetilde{w}\| < 2\widetilde{\eta} < \delta,$$

and hence $\Omega \cap (\bar{w} + \widetilde{\eta} I\!B) \subset E$. In this way we arrive at the desired subgradient inclusion $x^* \in \widehat{N}_{\varepsilon+\eta}(\bar{w}; \Omega)$. The remaining relation in (3.14) follows from the estimates

$$\begin{aligned}\|\bar{x} - \bar{w}\| &\leq \|\bar{x} - \widetilde{w}\| + \|\widetilde{w} - \bar{w}\| \leq \|F\| p_F(\widetilde{w} - \bar{x}) + \widetilde{\eta} \\ &\leq \|F\|\left(\tau_F(\bar{x}; \Omega) + \widetilde{\eta}^2\right) + \widetilde{\eta} \leq \|F\|\tau_F(\bar{x}; \Omega) + \widetilde{\eta}(\|F\| + 1) \\ &\leq \|F\|\tau_F(\bar{x}; \Omega) + \eta,\end{aligned}$$

which complete the proof of the theorem. □

We are now ready to establish major relations between *limiting subgradients* of minimal time functions at *in-set* points and *limiting normals* to the corresponding target sets.

Theorem 3.6 (relations between limiting subgradients of minimal time functions at in-set points and limiting normals to target sets). *Let $\bar{x} \in \Omega$ for the minimal time function* (1.1). *Then we have*

$$\partial\tau_F(\bar{x};\Omega) \subset N(\bar{x};\Omega) \cap \left\{x^* \in X^* \middle|\ p_{F^\circ}(-x^*) \le 1\right\}, \tag{3.17}$$

$$N(\bar{x};\Omega) = \bigcup_{\lambda>0} \lambda\partial\tau_F(\bar{x};\Omega). \tag{3.18}$$

Proof First we justify (3.17). Take any $x^* \in \partial\tau_F(\bar{x};\Omega)$ and by definition (2.5) for the continuous function (1.1) find sequences $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{\partial}_{\varepsilon_k}\tau_F(x_k;\Omega)$ for all $k \in I\!N$. If there is a subsequence of $\{x_k\}$ (without relabeling) that entirely belongs to Ω, then $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ and

$$p_{F^\circ}(-x_k^*) \le 1 + \varepsilon_k\|F\| \tag{3.19}$$

along this subsequence by Theorem 3.4. By passing to the limit in $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ as $k \to \infty$ and using definition (2.2) of limiting normals we arrive at the inclusion $x^* \in N(\bar{x};\Omega)$. Furthermore, the supremum representation of the dual Minkowski function $p_{F^\circ}(\cdot)$ presented in Proposition 2.1 (ii) implies by (3.19) the estimate

$$\langle -x_k^*, v\rangle \le 1 + \varepsilon_k\|F\| \quad \text{for all} \quad v \in F$$

for all $k \in I\!N$ along the afore-mentioned subsequence, which in turns gives $\langle -x^*, v\rangle \le 1$ by passing to the limit as $k \to \infty$, since $x_k^* \xrightarrow{w^*} x^*$ and $\varepsilon_k \downarrow 0$.

To establish inclusion (3.17) of the theorem, it remains to consider the case when $x_k \notin \Omega$ for all $k \in I\!N$ sufficiently large. Then the upper estimate in Proposition 3.2 and the results of Theorem 3.5 applied in this case ensure inequality (3.19) for all large $k \in I\!N$ and justify the existence of a sequence $\{w_k\} \subset \Omega$ along which we have the relations

$$x_k^* \in \widehat{N}_{\varepsilon_k+1/k}(w_k;\Omega) \quad \text{and} \quad \|x_k - w_k\| \le \|F\|\tau_F(x_k;\Omega) + 1/k, \quad k \in I\!N. \tag{3.20}$$

It follows from the second relation in (3.20) by the continuity of the minimal time function in Proposition 2.2 (ii) that $w_k \to \bar{x}$ as $k \to \infty$. Thus $x^* \in N(\bar{x};\Omega)$ by passing to the limit in the first relation of (3.20) as $k \to \infty$. The proof of the inequality $p_{F^\circ}(-x^*) \le 1$ in this case is the same as given above for the in-set case wit the use of Proposition 3.2.

Let us now justify the inclusion

$$N(\bar{x};\Omega) \subset \bigcup_{\lambda>0} \lambda\partial\tau_F(\bar{x};\Omega), \tag{3.21}$$

which implies equality (3.18), since the opposite inclusion immediately follows from the one in (3.17). Take any $x^* \in N(\bar{x};\Omega)$ and by definition (2.2) of limiting normals find $\varepsilon_k \downarrow 0$, $w_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(w_k;\Omega)$ for all $k \in I\!N$. In view of Proposition 2.1 (ii), consider

$$\lambda_k := p_{F^\circ}(-x_k^*) + 1 = \sup_{v\in F}\langle -x_k^*, v\rangle + 1 \ge 1, \quad k \in I\!N,$$

and observe that the sequence $\{\lambda_k\}$ is also bounded from above by the assumed boundedness of the set F in X and the boundedness of the sequence $\{x_k^*\}$ in X^* due the uniform boundedness principle. With no loss of generality we suppose that $\lambda_k \to \lambda > 0$ as $k \to \infty$. It follows from the second inclusion in Theorem 3.4 that

$$\frac{x_k^*}{\lambda_k} \in \widehat{\partial}_{\alpha\varepsilon_k} \tau_F(w_k; \Omega), \quad k \in I\!N, \tag{3.22}$$

with $\alpha = 2\|F\| \cdot \|F^\circ\| + 1$. Thus we get

$$x^* \in \lambda \partial \tau_F(\bar{x}; \Omega)$$

by passing to the limit in (3.22) as $k \to \infty$. This justifies inclusion (3.21) and hence the equality representation (3.18), which completes the proof of the theorem. □

The next result establishes relations between limiting subgradients of minimal time functions at *out-of-set* points and limiting normals to the corresponding target sets at points of the *minimum set* $M_F(\bar{x})$ defined in (3.1) under a certain *well-posedness* of the initial data in (1.1) formulated as follows.

Definition 3.7 (*well-posedness condition for minimal time functions*). We say that the WELL- POSEDNESS CONDITION holds for the minimal time (1.1) at $\bar{x} \notin \Omega$ if for any sequences $\varepsilon_k \downarrow 0$ and $x_k \to \bar{x}$ as $k \to \infty$ with $\widehat{\partial}\tau_F(x_k; \Omega) \neq \emptyset$, $k \in I\!N$, there is a sequence of minimum set points $w_k \in M_F(x_k)$ from (3.1) that contains a convergent subsequence.

We refer the reader to [13] and to [11, Sect. 1.3.3] for more discussions on this property and sufficient conditions for its validity in the case of distance functions.

Theorem 3.8 (limiting subgradients of minimal time functions at out-of-set points via minimum set points under well-posedness). *Let the well-posedness condition of Definition 3.7 be satisfied for the minimal time function (1.1) at some point $\bar{x} \notin \Omega$. Then the limiting subgradient inclusion*

$$\partial \tau_F(\bar{x}; \Omega) \subset \bigcup_{\bar{w} \in M_F(\bar{x}; \Omega)} \partial \tau_F\left(t\bar{w} + (1-t)\bar{x}; \Omega\right) \cap \left\{x^* \in X^* \mid p_{F^\circ}(-x^*) \le 1\right\}. \tag{3.23}$$

holds whenever $t \in (0, 1]$. In particular, we have the upper estimate

$$\partial \tau_F(\bar{x}; \Omega) \subset \bigcup_{\bar{w} \in M_F(\bar{x}; \Omega)} N(\bar{w}; \Omega) \cap \left\{x^* \in X^* \mid p_{F^\circ}(-x^*) \le 1\right\}. \tag{3.24}$$

Proof To justify (3.23), take any $x^* \in \partial \tau_F(\bar{x}; \Omega)$ and find by definition sequences

$$\varepsilon_k \downarrow 0, \quad x_k \to \bar{x}, \quad \text{and} \quad x_k^* \xrightarrow{w^*} x^* \quad \text{with} \quad x_k^* \in \widehat{\partial}_{\varepsilon_k} \tau_F(x_k; \Omega), \quad k \in I\!N. \tag{3.25}$$

By the well-posedness condition there is a subsequence of $w_k \in M_F(x_k)$ (without relabeling) converging to to some point $\bar{w}$. It is easy to observe from the standing closedness assumptions on F and Ω that $\bar{w} \in M_F(\bar{x})$. Employing now Proposition 3.8, we get

$$x_k^* \in \widehat{\partial}_{\varepsilon_k} \tau_F\left(t w_k + (1-t)x_k; \Omega\right) \quad \text{and} \quad p_{F^\circ}(-x_k^*) \le 1 + \varepsilon_k \|F\|$$

for all $t \in (0, 1]$ and $k \in I\!N$. The latter implies, by passing to the limit in (3.25) as $k \to \infty$ and repeating the corresponding arguments in the proof of Theorem 3.6, that

$$x^* \in \partial \tau_F\left(t\bar{w} + (1-t)\bar{x}; \Omega\right) \quad \text{and} \quad p_F(-x^*) \le 1,$$

and thus we have (3.23). Finally, (3.24) follows from (3.23) as $t = 1$ and inclusion (3.17) of Theorem 3.6. This completes the proof of the theorem. □

4 Subgradients of minimal time functions via normals to enlargement sets

In this section, we establish relations between ε-subgradients and limiting subgradients of the minimal time function (1.1) at *out-of-set* points $\bar{x} \notin \Omega$ and the corresponding normals to the following *enlargements*

$$\Omega_r := \{x \in X |\ \tau_F(x; \Omega) \le r\}, \quad r > 0, \tag{4.1}$$

of the target set Ω. First we present a useful improvement of [9, Lemma 3.5].

Lemma 4.1 (minimal time functions for enlargements of target sets). *For any $x \notin \Omega_r$ with $r > 0$ we have the relation*

$$\tau_F(x; \Omega_r) + r = \tau_F(x; \Omega). \tag{4.2}$$

Proof Fix $w \in \Omega$ and consider the function

$$h(t) := \tau_F\left(w + t(x - w); \Omega\right), \quad 0 \le t \le 1.$$

It is clear that $h(\cdot)$ is continuous on $[0, 1]$ with the properties $h(0) = \tau_F(w; \Omega) = 0$ and $h(1) = \tau_F(x) > r$. Applying the classical *intermediate value theorem* to the function $h(\cdot)$, we find $t \in (0, 1)$ such that $\tau_F(z; \Omega) = r$ for some $z := w + (1 - t)(x - w)$; observe that we replace t by $1 - t$, which also belongs to the interval $(0, 1)$. Therefore,

$$\begin{aligned} r + \tau_F(x; \Omega_r) &= \tau_F(z; \Omega) + \tau_F(x; \Omega_r) \le p_F(w - z) + p_F(z - x) \\ &= p_F\left((1 - t)(w - x)\right) + p_F\left(t(w - x)\right) \\ &= (1 - t)p_F(w - x) + tp_F(w - x) = p_F(w - x). \end{aligned}$$

This implies that $\tau_F(x; \Omega_r) + r \le \tau_F(x; \Omega)$, since $w \in \Omega$ was chosen arbitrarily. The opposite inequality in (4.2) is established in [9, Lemma 3.5]. Thus we get the equality in (4.2) and complete the proof of the lemma. □

The next theorem provides *two-sided* estimates of ε-subgradients of minimal time functions at out-of-set points of target sets via ε-normals to their enlargements (4.1). For $\varepsilon = 0$ it gives the *equality* representation for Fréchet subgradients of minimal time functions obtained in [9, Theorem 4.2]. In our opinion, there is a gap in the proof of the latter result in [9] related to an uncorrect application of [9, Lemma 3.5] to derive formula (4.10) in [9], which does not look to be fully correct and to complete the proof of theorem. Our proof presented below follows another route, which allows us to avoid this gap.

Theorem 4.2 (relations between ε-subgradients of minimal time functions at out-of-set points and ε-normals to enlargements of target sets). *Let $\bar{x} \notin \Omega$ in (1.1), and let $r := \tau_F(\bar{x}; \Omega)$. Then for any $\varepsilon \ge 0$ we have the two-sides estimates*

$$\begin{aligned} \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega) &\subset \widehat{N}_\varepsilon(\bar{x}; \Omega_r) \cap \left\{x^* \in X^* |\ 1 - \varepsilon\|F\| \le p_{F^\circ}(-x^*) \le 1 + \varepsilon\|F\|\right\} \\ &\subset \widehat{\partial}_{\alpha\varepsilon} \tau_F(\bar{x}; \Omega), \end{aligned} \tag{4.3}$$

where the perturbation parameter $\alpha = 2\|F\| \cdot \|F^\circ\| + 1$ is defined in Theorem 3.4. In particular, the Fréchet subdifferential of the minimal time function (1.1) is computed by

$$\widehat{\partial} \tau_F(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega_r) \cap \left\{x^* \in X^* |\ p_{F^\circ}(-x^*) = 1\right\}. \tag{4.4}$$

Proof Pick any $x^* \in \widehat{\partial}_\varepsilon \tau_F(\bar{x}; \Omega)$. Then Proposition 3.2 yields that

$$1 - \varepsilon\|F\| \leq p_{F^\circ}(-x^*) \leq 1 + \varepsilon\|F\|, \quad \varepsilon \geq 0. \tag{4.5}$$

Applying further definition (2.3) to the given Fréchet subgradient x^*, for any $\eta > 0$ find $\delta > 0$ such that (3.5) holds. Then for every $x \in \Omega_r \cap (\bar{x} + \delta I\!B)$, we get

$$\tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega) = \tau_F(x; \Omega) - r \leq 0$$

by the construction of Ω_r in (4.1). Substituting the latter into (3.5) gives

$$\langle x^*, x - \bar{x}\rangle \leq (\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for all} \quad x \in \Omega_r \cap (\bar{x} + \delta I\!B), \tag{4.6}$$

which means that $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r)$ by (2.2) and thus justifies the first inclusion in (4.3).

To prove the second inclusion in (4.3), fix $x^* \in \widehat{N}_\varepsilon(\bar{x}; \Omega_r)$ satisfying (4.5). Given any $\eta > 0$, find by definition (2.2) of ε-normals a number $\delta > 0$ such that (4.6) holds and also, by the second inclusion of Theorem 3.4, that

$$\langle x^*, x - \bar{x}\rangle \leq \tau_F(x; \Omega_r) + (\alpha\varepsilon + \eta)\|x - \bar{x}\| \quad \text{for all} \quad x \in \Omega_r \cap (\bar{x} + \delta I\!B), \tag{4.7}$$

where $\alpha = 2\|F\| \cdot \|F^\circ\| + 1$. Take now any $x \in X$ satisfying

$$x - \bar{x} \in \left(\frac{\delta}{1 + \|F\| \cdot \|F^\circ\|}\right) I\!B \tag{4.8}$$

and assume first that $x \notin \Omega_r$. By Lemma 4.1 we have

$$\tau_F(x; \Omega_r) = \tau_F(x; \Omega) - r = \tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega).$$

Substituting the latter into (4.7) gives the estimate

$$\langle x^*, x - \bar{x}\rangle \leq \tau_F(x; \Omega) - \tau_F(\bar{x}; \Omega) + (\alpha\varepsilon + \eta)\|x - \bar{x}\|. \tag{4.9}$$

Observe also that (4.9) holds trivially if $\tau_F(x; \Omega) = r$.

It remains to consider the case of $x \in \Omega_r$ in (4.8) with $\tau_F(x; \Omega) < r$. Put

$$t := r - \tau_F(x; \Omega) > 0$$

and, taking into account that $p_{F^\circ}(-x^*) \geq 1 - \varepsilon\|F\|$, for any $k \in I\!N$ find $w_k \in F$ such that $\langle -x^*, w_k\rangle \geq 1 - \varepsilon\|F\| - k^{-1}$. By Proposition 2.2(ii) we have the estimate

$$t = \tau_F(\bar{x}; \Omega) - \tau_F(x; \Omega) \leq \|F^\circ\| \cdot \|x - \bar{x}\|. \tag{4.10}$$

It easily follows from the choice of x and Proposition 2.2 (i) that

$$\tau_F(x - tw_k; \Omega) \leq \tau_F(x; \Omega) + p_F(tw_k) \leq \tau_F(x) + t \leq r,$$

since $p_F(w_K) \leq$ due to $w_k \in F$. By (4.8) and (4.10) we also have the estimates

$$\begin{aligned}\|x - tw_k - \bar{x}\| &\leq \|x - \bar{x}\| + t\|w_k\| \leq \|x - \bar{x}\| + \|F\| \cdot \|F^\circ\| \cdot \|x - \bar{x}\| \\ &\leq \left(1 + \|F\| \cdot \|F^\circ\|\right)\|x - \bar{x}\| \leq \delta.\end{aligned}$$

Substituting this into (4.6) allows us to conclude that

$$\begin{aligned}\langle x^*, x - tw_k - \bar{x}\rangle &\leq (\varepsilon + \eta)\|x - tw_k - \bar{x}\| \\ &\leq (\varepsilon + \eta)\left(1 + \|F\| \cdot \|F^\circ\|\right)\|x - \bar{x}\|,\end{aligned}$$

which implies by the above choice of w_k and definition of t that

$$\begin{aligned}\langle x^*, x-\bar{x}\rangle &\le t\langle x^*, w_k\rangle + (\varepsilon+\eta)\left(1+\|F\|\cdot\|F^\circ\|\right)\|x-\bar{x}\|\\ &\le t(-1+\varepsilon\|F\|+k^{-1}) + (\varepsilon+\eta)\left(1+\|F\|\cdot\|F^\circ\|\right)\|x-\bar{x}\|\\ &\le \tau_F(x;\Omega)-\tau_F(\bar{x};\Omega)+\left[(\varepsilon\|F\|+k^{-1})\|F^\circ\|+(\varepsilon+\eta)\left(1+\|F\|\cdot\|F^\circ\|\right)\right]\\ &\quad \|x-\bar{x}\|.\end{aligned}$$

Letting finally $k\to\infty$ and taking (4.9) into account and the fact that $\eta>0$ was chosen arbitrarily η, we have $x^*\in\widehat{\partial}_{\alpha\varepsilon}\tau_F(\bar{x};\Omega)$ with the number $\alpha>0$ defined above. Thus we justify the second estimate in (4.3) and complete the proof. □

Our next goal is to establish appropriate *out-of-set* counterparts of the limiting subgradient-normal relations of Theorem 3.6 with the replacement of the target set Ω by its *enlargements* Ω_r defined in (4.1). To proceed in this direction, we employ a certain *one-sided/outer* modification of the limiting subdifferential recently introduced in [13] and applied therein to the study of distance functions; see also [11, Sect. 1.3.3].

Definition 4.3 (*right-sided limiting subdifferential*). Given an extended-real-valued function $\varphi\colon X\to\overline{I\!R}$ finite at $\bar{x}$, define its RIGHT-SIDED LIMITING SUBDIFFERENTIAL at this point by the sequential Painlevé–Kuratowski outer limit

$$\partial_{\ge}\varphi(\bar{x}) := \mathop{\mathrm{Lim\,sup}}_{\substack{x\overset{\varphi+}{\to}\bar{x}\\ \varepsilon\downarrow 0}}\widehat{\partial}_{\varepsilon}\varphi(x), \tag{4.11}$$

where $x\overset{\varphi+}{\to}\bar{x}$ means that $x\to\bar{x}$ with $\varphi(x)\to\varphi(\bar{x})$ and $\varphi(x)\ge\varphi(\bar{x})$.

Observe that the only difference of $\partial_{\ge}\varphi(\bar{x})$ from our basic subdifferential construction (2.5) is that in (4.11) we involve into the limiting procedure those $x\to\bar{x}$ for which $\varphi(x)\to\varphi(\bar{x})$ *and* $\varphi(x)\ge\varphi(\bar{x})$ in contrast to all $x\overset{\varphi}{\to}\bar{x}$. We obviously have the inclusions

$$\widehat{\partial}\varphi(\bar{x})\subset\partial_{\ge}\varphi(\bar{x})\subset\partial\varphi(\bar{x}),$$

which may be *strict* while both hold as equalities when, in particular, φ is *lower regular* at $\bar{x}$; this encompasses convex, amenable, and other classes of "nice" functions; see e.g., [11].

Theorem 4.4 (relations between right-sided limiting subgradients of minimal time functions at out-of-set points and limiting normals to enlargements of target sets). *Let $\bar{x}\notin\Omega$ for the minimal time function* (1.1)*, and let $r:=\tau_F(\bar{x};\Omega)$. Then we have the inclusion*

$$\partial_{\ge}\tau_F(\bar{x};\Omega)\subset N(\bar{x};\Omega_r)\cap\left\{x^*\in X^*\,|\,p_{F^\circ}(-x^*)\le 1\right\} \tag{4.12}$$

and, furthermore, the equality representation

$$N(\bar{x};\Omega_r)=\bigcup_{\lambda\ge 0}\lambda\partial_{\ge}\tau_F(\bar{x};\Omega) \tag{4.13}$$

with the convention that $0\times\emptyset=0$.

Proof To justify inclusion (4.12), pick any $x^*\in\partial_{\ge}\tau_F(\bar{x};\Omega)$ and, by Definition 4.11 in the setting under consideration, find sequences $\varepsilon_k\downarrow 0$, $x_k\to\bar{x}$, and $x_k^*\overset{w^*}{\longrightarrow}x^*$ as $k\to\infty$ such that $\tau_F(x_k;\Omega)\ge r$ and $x_k^*\in\widehat{\partial}_{\varepsilon_k}\tau_F(x_k;\Omega)$ for all $k\in I\!N$. The proof of the inclusion $p_{F^\circ}(-x^*)\le 1$ is similar to the corresponding arguments in the proof of Theorem 3.6.

Let us show that x^* is a limiting normal to the enlargement Ω_r. Assume first that $\tau_F(x_k;\Omega) = r$ along a subsequence of $k \in I\!N$ (without relabeling). Then we have $x_k \in \Omega_r$ and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega_r)$; thus $x^* \in N(\bar{x};\Omega_r)$ by passing to the limit as $k \to \infty$.

Consider next the case when $\tau_F(x_k;\Omega) > r$ for all $k \in I\!N$ sufficiently large. Denote $\eta_k := \tau_F(x_k;\Omega) - r > 0$ and observe that $\eta_k \downarrow 0$ as $k \to \infty$ by the continuity of $\tau_F(\cdot;\Omega)$. It follows from Lemma 4.1 that

$$\widehat{\partial}_\varepsilon \tau_F(x;\Omega) = \widehat{\partial}_\varepsilon \tau_F(x;\Omega_r) \quad \text{whenever} \quad x \notin \Omega_r.$$

Applying Theorem 3.5 along the sequence of triples $\{x_k, \varepsilon_k, \eta_k\}$, find $w_k \in \Omega_r$ such that

$$x_k^* \in \widehat{N}_{\varepsilon_k+\eta_k}(w_k;\Omega_r) \quad \text{and} \quad \|x_k - w_k\| \le \|F\|\tau_F(x_k;\Omega_r) + \eta_k, \quad k \in I\!N.$$

This gives $x^* \in N(\bar{x};\Omega_r)$ by passing to the limit as $k \to \infty$ and thus justifies (4.12).

The latter inclusion clearly implies the one of "$\supset$" in (4.13). To justify the opposite inclusion in (4.13), take any $x^* \in N(\bar{x};\Omega_r)$ and find by definition sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega_r} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega_r)$ such that $x_k^* \xrightarrow{w^*} x^*$ as $k \to \infty$. Denote

$$\lambda_k := p_{F^\circ}(-x_k^*) + 1, \quad k \in I\!N,$$

and observe that the sequence $\{\lambda_k\}$ is bounded by Proposition 2.1(ii), the boundedness of F and of $\{x_k^*\}$ due to the uniform boundedness principle. Hence $\lambda_k \to \lambda \in [1,\infty)$ along a subsequence of $k \in I\!N$, with no relabeling. If $x^* = 0$ for the above weak* limit, than it belongs to the right-hand side of (4.13) by the convention that $0 \times \emptyset = 0$.

It remains to consider the case of $x^* \ne 0$. Observe that in this case $\tau_F(x_k;\Omega) = r$ when k is sufficiently large. Indeed, the opposite assumption on $\tau_F(x_k;\Omega) < r$ gives by the continuity of $\tau_F(\cdot;\Omega)$ that $x_k \in \operatorname{int}\Omega_r$, which easily implies that $\|x_k^*\| \le \varepsilon_k$ and hence

$$\|x^*\| \le \liminf_{k\to\infty} \|x_k^*\| = 0,$$

a contradiction. Thus we have from the the constructions of x_k^* and λ_k that

$$\widetilde{x}_k^* := \frac{x_k^*}{\lambda_k} \in \widehat{N}_{\varepsilon_k/\lambda_k}(x_k;\Omega_r) \quad \text{and} \quad p_{F^\circ}(-\widetilde{x}_k^*) \le 1, \quad k \in I\!N.$$

This implies by the second estimate in Theorem 4.2 for the set Ω_r that $\widetilde{x}_k^* \in \widehat{\partial}_{\alpha\varepsilon_k/\lambda_k}\tau_F$ for all $k \in I\!N$ with $\alpha = 2\|F\|\cdot\|F^\circ\| + 1$. By passing to the limit as $k \to \infty$ and employing Definition 4.11, we finally arrive at the inclusion

$$\frac{x^*}{\lambda} \in \partial_{\ge}\tau_F(\bar{x};\Omega),$$

which yields $x^* \in \lambda\partial_{\ge}\tau_F(\bar{x};\Omega)$ and completes the proof of theorem. □

5 Relations between subgradients of minimal time and Minkowski functions

In the previous sections, we established various results on relations between ε-subgradients and limiting subgradients of the minimal time function (1.1) and the corresponding normals to the target set Ω and its enlargements Ω_r. These results involve the Minkowski function (1.2) of the polar F° to the velocity set F in (1.1). The main goal of the concluding section of the paper is to derive new relations between ε-subgradients and limiting subgradients of the (generally nonconvex) minimal time function (1.1) at out-of-set points and the corresponding subgradients of (always convex) Minkowski function (1.2) involving points of the minimum (generalized projection) set $M_F(\bar{x};\Omega)$ from (3.1).

We start with the following proposition, which is a certain counterpart of Proposition 3.2 with the replacement of the Minkowski function to the polar set F° by this function to the velocity set itself taken at some minimum points of (3.1).

Proposition 5.1 (relations between ε-subgradients of minimal time functions at out-of-set sets and values of the corresponding Minkowski function at minimum set points). *Let $\bar{x}\notin\Omega$, let $\bar{w}\in M_F(\bar{x};\Omega)$, and let $\varepsilon\ge 0$. Then for any $x^*\in\widehat{\partial}_\varepsilon\tau(\bar{x};\Omega)$ we have the estimates*

$$1-\varepsilon\|F\|\le\left\langle -x^*,\frac{\bar{w}-\bar{x}}{p_F(\bar{w}-\bar{x})}\right\rangle\le 1+\varepsilon\|F\|.$$

Proof Since $x^*\in\widehat{\partial}_\varepsilon\tau(\bar{x};\Omega)$ and $(\bar{w}-\bar{x})/p_F(\bar{w}-\bar{x})]^{-1}\in F$, it follows from Proposition 5.1(ii) and the second estimate of Proposition 3.2 that

$$\left\langle -x^*,\frac{\bar{w}-\bar{x}}{p_F(\bar{w}-\bar{x})}\right\rangle\le p_{F^\circ}(-x^*)\le 1+\varepsilon\|F\|.$$

Furthermore, arguing as in the proof of the first estimate in Proposition 3.2, with $w_t:=\bar{w}$ therein, we conclude that

$$1-\varepsilon\|F\|\le\left\langle x^*,\frac{\bar{w}-\bar{x}}{p_F(\bar{w}-\bar{x})}\right\rangle,$$

which completes the proof of this proposition. □

The next result provides an *upper estimate* of the ε-subdifferential (2.3) of the minimal time function (1.1) at out-of-set points via the ε-subdifferential (2.4) of the *convex* Minkowski function (1.2).

Theorem 5.2 (relation between $\varepsilon-$subgradients of minimal time and Minkowski functions). *Let $\varepsilon\ge 0$, let $\bar{w}\in M_F(\bar{x})$, and let $\varepsilon\ge 0$. Then the upper estimate*

$$\begin{aligned}\widehat{\partial}_\varepsilon\tau_F(\bar{x};\Omega)\subset\Big\{x^*\in\widehat{N}_\varepsilon(\bar{w};\Omega)|\ \langle x^*,\widetilde{x}-x\rangle\le p_F(x;\Omega)-p_F(\widetilde{x};\Omega)\\ +\varepsilon\|x-\widetilde{x}\|\quad\text{for all}\quad x\in X\Big\}\end{aligned}\tag{5.1}$$

is satisfied, where $\widetilde{x}:=\bar{w}-\bar{x}$. In particular, for $\varepsilon=0$ we have

$$\widehat{\partial}\tau_F(\bar{x};\Omega)\subset\Big\{x^*\in\widehat{N}(\bar{w};\Omega)|\ \langle x^*,\widetilde{x}-x\rangle\le p_F(x;\Omega)-p_F(\widetilde{x};\Omega)\quad\text{for all}\quad x\in X\Big\}$$

Proof We begin with the observation that

$$\widehat{\partial}_\varepsilon\tau_F(\bar{x};\Omega)\subset\widehat{N}_\varepsilon(\bar{w};\Omega)$$

by Proposition 3.3 for $t=1$ and the first inclusion of Theorem 3.4. To proceed, fix any $x^*\in\widehat{\partial}_\varepsilon\tau_F(\bar{x};\Omega)$ and, given $\eta>0$, find $\delta>0$ such that

$$\langle x^*,x-\bar{x}\rangle\le\tau_F(x;\Omega)-\tau_F(\bar{x};\Omega)+(\varepsilon+\eta)\|x-\bar{x}\|\quad\text{whenever}\quad x\in\bar{x}+\delta I\!B.$$

It surely holds for every $x\in X$ that $\bar{x}-t(x-\widetilde{x})\in\bar{x}+\delta I\!B$ if $t>0$ is sufficiently small. Then for such $t>0$, by using the afore-mentioned properties of the Minkowski function, we have the following relations satisfied whenever $x \in X$:

$$\begin{aligned}\langle x^*, -t(x-\widetilde{x})\rangle &\le \tau_F(\bar{x}-t(x-\widetilde{x});\Omega)-\tau_F(\bar{x};\Omega)+(\varepsilon+\eta)t\|x-\widetilde{x}\|\\ &\le p_F\,(\bar{w}-\bar{x}+t(x-\widetilde{x}))-p_F(\bar{w}-\bar{x})+(\varepsilon+\eta)t\|x-\widetilde{x}\|\\ &= p_F\,(tx+(1-t)(\bar{w}-\bar{x}))-p_F(\bar{w}-\bar{x})+(\varepsilon+\eta)t\|x-\widetilde{x}\|\\ &\le tp_F(x)+(1-t)p_F(\bar{w}-\bar{x})-p_F(\bar{w}-\bar{x})+(\varepsilon+\eta)t\|x-\widetilde{x}\|\\ &= tp_F(x)-tp_F(\bar{w}-\bar{x})+(\varepsilon+\eta)t\|x-\widetilde{x}\|\\ &= tp_F(x)-tp_F(\widetilde{x})+(\varepsilon+\eta)t\|x-\widetilde{x}\|.\end{aligned}$$

Dividing both sides of the latter estimate by t and passing to the limit as $\eta \downarrow 0$, we arrive at (5.1) and complete the proof of the theorem. □

Our final result establishes an upper estimate for the limiting subdifferential (2.5) of the minimal time function under consideration via the classical subdifferential of the convex Minkowski function (1.2).

Theorem 5.3 (limiting subgradients of minimal time functions at out-of-set points via subgradients of convex analysis). *Let $\bar{x} \notin \Omega$, and let the well-posedness condition of Definition 3.7 be satisfied for the minimal time function (1.1) at $\bar{x}$. Then we have*

$$\partial\tau_F(\bar{x};\Omega)\subset\bigcup_{w\in M(\bar{x})}\big\{x^*\in N(w;\Omega)\,|\,\langle x^*, w-\bar{x}-x\rangle\le p_F(x;\Omega)-p_F(w-\bar{x};\Omega),\ \ x\in X\big\}.$$

Proof Take any $x^* \in \partial\tau_F(\bar{x};\Omega)$ and by definition (2.5) find sequences $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, and $x_k^* \in \widehat{\partial}_{\varepsilon_k}\tau_F(x_k;\Omega)$ as $k \to \infty$ such that $x_k^* \xrightarrow{w^*} x^*$ and

$$x_k^* \in \widehat{\partial}_{\varepsilon_k}\tau_F(x_k;\Omega) \quad \text{for all} \quad k \in I\!N. \tag{5.2}$$

By the assumed well-posedness property of (1.1), select a subsequence of $w_k \in M_F(x_k;\Omega)$ converging to some $\bar{w}$. It easily follows from the standing closedness assumptions that $\bar{w} \in M_F(\bar{x};\Omega)$. Applying Theorem 5.2 to all the inclusion in (5.2), we have

$$\begin{aligned}\widehat{\partial}_{\varepsilon_k}\tau_F(x_k;\Omega)\subset\big\{x^*\in\widehat{N}_{\varepsilon_k}(w_k;\Omega)|\ \langle x^*,\widetilde{x}_k-x\rangle &\le p_F(x;\Omega)-p_F(\widetilde{x}_k;\Omega)\\ &+\varepsilon_k||x-\widetilde{x}_k|| \quad \text{whenever} \quad x\in X\}\end{aligned}$$

along this subsequence, where $\widetilde{x}_k := w_k - x_k$. This implies the inclusion claimed in the theorem by passing to the limit as $k \to \infty$ due to the constructions of limiting normals and subgradients, which thus completes the proof. □

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